Lower bounds on maximal determinants of ± 1 matrices via the probabilistic method

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December 4, 2012

Abstract

We show that the maximal determinant $D(n)$ for $n \times n$ $\{\pm 1\}$-matrices satisfies $\mathcal{R}(n) := D(n)/n^{n/2} \geq \kappa_d > 0$. Here $n^{n/2}$ is the Hadamard upper bound, and κ_d depends only on $d := n - h$, where h is the maximal order of a Hadamard matrix with $h \leq n$. Previous lower bounds on $\mathcal{R}(n)$ depend on both d and n. Our bounds are improvements, for all sufficiently large n, if $d > 1$.

We give various lower bounds on $\mathcal{R}(n)$ that depend only on d. For example, $\mathcal{R}(n) \geq 0.07\,(0.352)^d > 3^{-(d+3)}$. For any fixed $d \geq 0$ we have $\mathcal{R}(n) \geq (2/(\pi e))^{d/2}$ for all sufficiently large n (and conjecturally for all positive n). If the Hadamard conjecture is true, then $d \leq 3$ and $\kappa_d \geq (2/(\pi e))^{d/2} > 1/9$.

1 Introduction

Let $D(n)$ be the maximal determinant possible for an $n \times n$ matrix with elements drawn from the real interval $[-1, 1]$. Hadamard [32][1] proved that $D(n) \le n^{n/2}$, and the *Hadamard conjecture* is that a matrix achieving this upper bound exists for each positive integer n divisible by four. The function $\mathcal{R}(n) := D(n)/n^{n/2}$ is a measure of the sharpness of the Hadamard bound. Clearly $\mathcal{R}(n) = 1$ if a Hadamard matrix of order n exists; otherwise $\mathcal{R}(n) < 1$. The aim of this paper is to give lower bounds on $\mathcal{R}(n)$.

If $h \le n$ is the order of a Hadamard matrix, and $d = n - h$, then we show that $\mathcal{R}(n)$ is bounded below by a positive constant κ_d (depending on d but not on n). When $d > 1$ this improves on previous results[2] for which the lower bound was (at best) of order $n^{-\alpha d}$ for some constant $\alpha \ge 1/2$. Rokicki *et al* [50] conjectured that $\mathcal{R}(n) \ge 1/2$ on the basis of computational results for $n \le 120$.

We obtain lower bounds on $\mathcal{R}(n)$ using the probabilistic method pioneered by Erdős (see for example [2, 29]). Specifically, we adjoin d extra columns to the $h \times h$ Hadamard matrix, and fill their $h \times d$ entries with random signs obtained by independently tossing fair coins. Then we adjoin d extra rows, and fill their $d \times (h+d)$ entries with ± 1 signs chosen deterministically in a way intended to approximately maximize the determinant of the final matrix. To do so, we use the fact that this determinant can be expressed in terms of the $d \times d$ Schur complement (see §3). In the proof of Theorem 1 we obtain a lower bound on the expected value of the determinant in a direct manner. In the proofs of Theorems 2 and 3 we use a Hoeffding tail bound to show that the Schur complement is, with high probability, sufficiently diagonally dominant that its determinant is close to the product of its diagonal elements. We employ two possibly new inequalities, Lemma 8 and Lemma 10 in §4, that give lower bounds on the determinant of a diagonally dominant matrix. The bounds are sharper than the obvious bounds arising from Gerschgorin's circle theorem [31, 59], so may be of independent interest.

In the special case $d = 1$ our argument simplifies, because there is no need to consider a nontrivial Schur complement or to deal with the contribution of the off-diagonal elements. This case was (essentially) already considered by

[1]For earlier contributions by Desplanques, Lévy, Muir, Sylvester and Thomson (Lord Kelvin), see [44, 56] and [42, pg. 384].

[2]See [10, Theorem 1] and the references cited there. For example, the well-known bound of Clements and Lindström [13, Corollary to Thm. 2] only shows that $\mathcal{R}(n) > (3/4)^{n/2}$.

Brown and Spencer [12], Erdős and Spencer [29, Ch. 15], and (independently) by Best [8]; see also [2, §2.5] and [3, Problem A4]. The consequence for lower bounds on $\mathcal{R}(n)$ when $n \equiv 1 \bmod 4$ was exploited by Farmakis and Kounias [30], and an improvement using 3-normalized Hadamard matrices was considered by Orrick and Solomon [47].

In §2 we review previous results that give upper bounds on gaps between the orders of Hadamard matrices. These are relevant as they enable us to bound $d = n - h$ as a function of h.

Various preliminary results are proved in §4, and the main results are proved in §5. Theorem 1 applies for fixed d and $h \geq h_0(d)$, where the function $h_0(d)$ grows rapidly, but this is not significant for the cases $d \leq 3$ that arise if we assume the Hadamard conjecture. For $d \leq 3$, Corollary 1 shows that $\mathcal{R}(n)$ is bounded below by $(2/(\pi e))^{d/2} > 1/9$, coming close to Rokicki *et al*'s conjectured lower bound of $1/2$, and improving on earlier results [10, 13, 14, 39, 40] that failed to obtain a constant lower bound on $\mathcal{R}(n)$ for $2 \leq d \leq 3$.

At the cost of more complicated proofs, Theorems 2 and 3 apply to larger regions of (d, h)-space. Theorem 2 applies for $h/\ln h \geq 16d^3$, and Theorem 3 applies for $h \geq 6d^3$. In view of known results on gaps between Hadamard orders, discussed in §2, these theorems give a lower bound on $\mathcal{R}(n)$ for all but a finite set E of positive integers n. We have obtained a lower bound on $\mathcal{R}(n)$ for each $n \in E$ by explicit computation, using a probabilistic algorithm that uses the same construction as the proofs of these theorems. This leads to Theorem 4, which gives a lower bound $\mathcal{R}(n) > 3^{-(d+3)}$ that is valid for all positive integers n (the constants here are not the best possible).

Acknowledgements

We thank Robert Craigen for informing us of the work of his student Ivan Livinskyi, and Will Orrick for his comments and for providing a copy of the unpublished report [50]. Dragomir Djoković and Ilias Kotsireas shared their list of known small Hadamard orders, which was very useful for checking the program that we used in the proof of Lemma 15.

2 Gaps between Hadamard orders

In order to apply our results to obtain a lower bound on $\mathcal{R}(n)$ for given n, we need to know the order h of a Hadamard matrix with $h \le n$ and $n - h$ preferably as small as possible. Thus, it is of interest to consider the size of possible gaps in the sequence $(n_i)_{i \ge 1}$ of Hadamard orders. We define the *Hadamard gap function* $\gamma : \mathbb{R} \to \mathbb{Z}$ by

$$\gamma(x) := \max\{n_{i+1} - n_i \mid n_i \le x\} \cup \{0\}\,. \tag{1}$$

In [10] it was shown, using the Paley and Sylvester constructions, that $\gamma(n)$ can be bounded using the prime-gap function. For example, if p is an odd prime, then $2(p+1)$ is a Hadamard order. However, only rather weak bounds on the prime-gap function are known. A different approach which produces asymptotically-stronger bounds employs results of Seberry [60], as subsequently sharpened by Craigen [17], Livinskyi [41], and Smith [55]. These results take the following form: for any odd positive integer q, a Hadamard matrix of order $2^t q$ exists for every integer

$$t \ge \alpha \log_2(q) + \beta,$$

where α and β are author-dependent constants. Seberry [60] obtained $\alpha = 2$. Craigen [17] improved this to $\alpha = 2/3$, $\beta = 16/3$, and later obtained $\alpha = 3/8$ in unpublished work with Tiessen quoted in [37, Thm. 2.27] and [18, 21].[3] Livinskyi [41] found $\alpha = 1/5$, $\beta = 64/5$. Smith's unpublished paper [55] shows that $\gamma(n) = O(n^\varepsilon)$ for each $\varepsilon > 0$, but the constants hidden in the "O" in this result can be very large, so we do not use Smith's result here.

The connection between these results and the Hadamard gap function is given by Lemma 1. From the lemma and the results of Livinskyi, the Hadamard gap function satisfies

$$\gamma(n) = O(n^{1/6}). \tag{2}$$

This is much sharper than $\gamma(n) = O(n^{21/40})$ arising from the best current result for prime gaps (by Baker, Harman and Pintz [4]), although not as sharp as the result $\gamma(n) = O(\log^2 n)$ that would follow from Cramér's prime-gap conjecture [10, 23, 53, 54].

[3] There are typographical errors in [37, Thm. 2.27] and in [21, Thm. 1.43], where the floor function should be replaced by the ceiling function. This has the effect of increasing the additive constant β.

Lemma 1. *Suppose there exist constants α, β such that, for any odd positive integer q, a Hadamard matrix of order $2^t q$ exists for all $t \geq \alpha \log_2(q) + \beta$. Then the Hadamard gap function $\gamma(n)$ satisfies*

$$\gamma(n) = O(n^{\alpha/(1+\alpha)}) .$$

Proof. Consider consecutive odd integers q_0, $q_1 = q_0 + 2$ and corresponding $n_i = 2^t q_i$, where $t = \lceil \alpha \log_2(q_1) + \beta \rceil$. By assumption there exist Hadamard matrices of orders n_0, n_1. Also, $2^\beta q_1^\alpha \leq 2^t < 2^{\beta+1} q_1^\alpha$. Thus

$$n_1 = 2^t q_1 \geq 2^\beta q_1^{1+\alpha}$$

and $$n_1 - n_0 = 2^{t+1} < 2^{\beta+2} q_1^\alpha \leq 2^{2+\beta/(1+\alpha)} n_1^{\alpha/(1+\alpha)} = O(n_0^{\alpha/(1+\alpha)}).$$ □

3 The Schur complement

Let

$$\widetilde{A} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

be an $n \times n$ matrix written in block form, where A is $h \times h$, and $n = h + d > h$. Then the *Schur complement* [51] of A in $\widetilde{A}$ is the $d \times d$ matrix

$$D - CA^{-1}B.$$

The Schur complement is relevant to our problem due to the following lemma.

Lemma 2. *If $\widetilde{A}$ is as above, with A nonsingular, then*

$$\det(\widetilde{A}) = \det(A) \det(D - CA^{-1}B).$$

Proof. Using block Gaussian elimination on $\widetilde{A}$ gives

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix} .$$

Now take determinants. □

4 Notation and auxiliary results

In this section we define our notation and prove some auxiliary results that are needed in §5. As above, $D(n)$ is the maximum determinant function and $\mathcal{R}(n) := D(n)/n^{n/2}$ is its normalization by the Hadamard bound $n^{n/2}$. The set of orders of all Hadamard matrices is denoted by $\mathcal{H}$.

We define $c := \sqrt{2/\pi} \approx 0.7979$. Other constants are denoted c_1, c_2, α, β, etc. Usually $h \in \mathcal{H}$ and $n = h + d$, where $d \geq 0$ (the case $d = 0$ is trivial because then the Hadamard bound applies). We assume $h \geq 4$ to avoid the cases $h \in \{1, 2\}$, although in most cases it is easy to verify that the results also hold for $h \in \{1, 2\}$.

Matrices are denoted by capital letters A etc, and their elements by the corresponding lower-case letters, e.g. a_{ij} (the comma between subscripts is omitted if the meaning is clear).

When using the probabilistic method, the probability of an event S (which is always a discrete set of possible outcomes of a random process) is denoted by $\Pr(S)$, and the expectation of a random variable X is denoted by $E(X)$.

Lemma 3. *Suppose that h is an even positive integer.Then*

$$\binom{h}{h/2} > 2^h \sqrt{\frac{2}{\pi h}} \left(1 - \frac{1}{4h}\right).$$

Proof. This follows from Stirling's asymptotic expansion of $\ln \Gamma(x)$ with the error bounded by the first term omitted, see for example [11, eqn. (4.38)]. □

Lemma 4. *Let $g(h) := 1 + 2^{-h} h \binom{h}{h/2}$, where $h \geq 4$ is an even integer. Then $g(h) > ch^{1/2} + 1 - ch^{-1/2}/4$ and $g(h) > ch^{1/2} + 0.9$, where $c = \sqrt{2/\pi}$.*

Proof. The first inequality follows from Lemma 3. From the condition $h \geq 4$, we have $ch^{-1/2}/4 < 1/10$. Thus $g(h) > ch^{1/2} + 1 - 0.1 = ch^{1/2} + 0.9$. □

Lemma 5 is from [10, Lemma 2], and Lemma 6 is similar.

Lemma 5. *If $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$, $n > |\alpha| > 0$, and $h = n - \alpha$, then*

$$\frac{h^h}{n^n} > \left(\frac{1}{ne}\right)^{\alpha}.$$

Proof. Taking logarithms, and writing $x = \alpha/n$, the inequality reduces to

$$(1-x)\ln(1-x) + x > 0, \tag{3}$$

or equivalently (since $0 < |x| < 1$)

$$\frac{x^2}{1\cdot 2} + \frac{x^3}{2\cdot 3} + \frac{x^4}{3\cdot 4} + \cdots > 0.$$

This is clear if $x > 0$, and also if $x < 0$ because then the terms alternate in sign and decrease in magnitude. $\square$

Lemma 6. *If $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$, $n > |\alpha| > 0$, and $h = n - \alpha$, then*

$$(h/n)^n > \exp(-\alpha - \alpha^2/h).$$

Proof. Taking $x = \alpha/n$, the inequality (3) proved above implies that $\ln(1-x) > -x/(1-x)$, so

$$(1-x)^n \; > \; \exp\left(-\frac{nx}{1-x}\right).$$

Since $1 - x = h/n$, we obtain

$$\left(\frac{h}{n}\right)^n \; > \; \exp\left(-\frac{\alpha}{1-\alpha/n}\right) = \exp(-\alpha - \alpha^2/h).$$

$\square$

Lemma 7. *Let $A \in \{\pm 1\}^{h\times h}$ be a Hadamard matrix, $C \in \{\pm 1\}^{d\times h}$, and $U = CA^{-1}$. Then, for each i with $1 \le i \le d$,*

$$\sum_{j=1}^{h} u_{ij}^2 = 1.$$

Proof. Since A is Hadamard, $A^TA = hI$. Thus $UU^T = h^{-1}CC^T$. Since $c_{ij} = \pm 1$, $\mathrm{diag}(CC^T) = hI$. Thus $\mathrm{diag}(UU^T) = I$. $\square$

Definition 1. *If $A \in \mathbb{R}^{d\times d}$ satisfies $|a_{ij}| \le \varepsilon |a_{ii}|$ for all $i \ne j$, then we say that A is* $\mathrm{DD}(\varepsilon)$. (Here "DD" stands for "diagonally dominant".)

Lemma 8. *If* $A = I - E \in \mathbb{R}^{d\times d}$, $|e_{ij}| \le \varepsilon$ *for* $1 \le i,j \le d$, *and* $d\varepsilon \le 1$, *then*

$$\det(A) \ge 1 - d\varepsilon.$$

Proof. We first assume that $d\varepsilon < 1$. Thus, by Gerschgorin's theorem, A is nonsingular. Hence by continuity $\det(A) > 0$. Thus, $\ln\det(A)$ is well-defined and real. Write the eigenvalues of $X \in \mathbb{R}^{d\times d}$ as $\lambda_i(X) \in \mathbb{C}$, and define the *trace* $\mathrm{Tr}(X) := \sum_i x_{ii} = \sum_i \lambda_i(X)$. Then

$$\ln\det(A) = \ln\left(\prod_{i=1}^{d} \lambda_i(A)\right) = \mathrm{Tr}(\ln(A)),$$

where

$$\ln(A) = \ln(I - E) = -\sum_{k=1}^{\infty} \frac{1}{k} E^k\,.$$

Thus

$$\ln\det(A) = -\mathrm{Tr}\left(\sum_{k=1}^{\infty} \frac{1}{k} E^k\right) = -\sum_{k=1}^{\infty} \frac{1}{k}\mathrm{Tr}(E^k)\,.$$

Considering this series term by term, it is clear that $\mathrm{Tr}(E^k)$ attains its maximum value, subject to the constraints $|e_{ij}| \le \varepsilon$, when each $e_{ij} = \varepsilon$, that is when $E = E_1 := \varepsilon\, ee^T$, where $e^T := (1,1,\ldots,1)$ is the d-vector of all ones. Using $ee^T = d$, it is easy to prove, by induction on k, that $E_1^k = (d\varepsilon)^{k-1}E_1$ for all $k \ge 1$. Thus $\mathrm{Tr}(E_1^k) = (d\varepsilon)^{k-1}\mathrm{Tr}(E_1) = (d\varepsilon)^k$. So we have

$$\ln\det(A) \ge -\sum_{k=1}^{\infty} \frac{(d\varepsilon)^k}{k} = \ln(1 - d\varepsilon)\,,$$

and it follows that $\det(A) \ge 1 - d\varepsilon$. This completes the proof for $d\varepsilon < 1$. If $d\varepsilon = 1$ then $\det(A) \ge 0$ by a continuity argument. $\square$

Remark 1. It is easy to show, using a rank-1 updating formula, that

$$\det(I - \varepsilon\, ee^T) = 1 - d\varepsilon\,.$$

Thus, the inequality of Lemma 8 is best possible. One may see from the proof of Lemma 8 that if $\varepsilon > 0$ then tightness occurs only for $E = \varepsilon\, ee^T$. In this unique extreme case, the eigenvalues of $A = I - E$ are $1 - d\varepsilon$ (with multiplicity 1) and 1 (with multiplicity $d - 1$).

Remark 2. Gerschgorin's theorem gives $|\lambda_i(A) - 1| \le d\varepsilon$, but this only implies the much weaker inequality $\det(A) \ge (1 - d\varepsilon)^d$.

If, in addition to the conditions of Lemma 8, we assume that $e_{ii} = 0$, then in the extreme case the eigenvalues of A are all shifted up by ε. Thus we obtain the following lemma. The proof is omitted since it is similar to the proof of Lemma 8.

Lemma 9. *If $A = I - E \in \mathbb{R}^{d\times d}$, $|e_{ij}| \le \varepsilon$ for $1 \le i, j \le d$, $e_{ii} = 0$ for $1 \le i \le d$, and $(d-1)\varepsilon \le 1$, then*

$$\det(A) \ge (1 - (d-1)\varepsilon)\,(1+\varepsilon)^{d-1}.$$

The following lemma, which may be of independent interest, gives a lower bound on the determinant of a diagonally dominant matrix.

Lemma 10. *If $A \in \mathbb{R}^{d\times d}$ is $\mathrm{DD}(\varepsilon)$, then*

$$|\det(A)| \ge \left(\prod_{i=1}^{d} |a_{ii}|\right) \left(1 - (d-1)^2\varepsilon^2\right).$$

Proof. If $\varepsilon < 0$ then $A = 0$ and the result is trivial; if $(d-1)\varepsilon \ge 1$ then the inequality is trivial as the right side is not positive. Hence, assume that $0 \le (d-1)\varepsilon < 1$. If any $a_{ii} = 0$ then the result is trivial. Otherwise, apply Lemma 9 to SA, where $S = \mathrm{diag}(a_{ii}^{-1})$. Since $\det(A) = \det(SA)\prod_i a_{ii}$ and

$$(1-(d-1)\varepsilon)(1+\varepsilon)^{d-1} \ge (1-(d-1)\varepsilon)(1+(d-1)\varepsilon) = 1-(d-1)^2\varepsilon^2,$$

the corollary follows. $\square$

Remark 3. Lemma 10 is much sharper than the bound

$$|\det(A)| \ge \left(\prod_{i=1}^{d} |a_{ii}|\right) (1-(d-1)\varepsilon)^d$$

that follows from Gerschgorin's theorem. For example, if $a_{ii} = 1$ for $1 \le i \le d$ and $(d-1)\varepsilon = 1/2$, then Lemma 10 gives the lower bound $3/4$ whereas Gerschgorin's theorem gives 2^{-d}.

Lemma 11. *If* $\kappa, \varepsilon_0 \in \mathbb{R}$, $\varepsilon_0 > 0$, $|\kappa\varepsilon_0| < 1$, *then* $1 + \kappa\varepsilon \geq \exp(\beta\varepsilon)$ *for all* $\varepsilon \in [0, \varepsilon_0]$, *where*

$$\beta = \frac{\ln(1 + \kappa\varepsilon_0)}{\varepsilon_0}\,.$$

Proof. This follows from the concave-up nature of $\exp(K\varepsilon)$, and the fact that $1 + \kappa\varepsilon = \exp(\beta\varepsilon)$ at the two endpoints $\varepsilon = 0$ and $\varepsilon = \varepsilon_0$. $\square$

The following lemma is essentially Erdős and Spencer [29, Lemma 15.2], so we omit the (straightforward) proof.

Lemma 12. *If* $X \in [0, 1]$ *is a random variable with* $E(X) = \mu$, *then for* $\lambda < \mu$ *we have*

$$\Pr(X \geq \lambda) \geq \frac{\mu - \lambda}{1 - \lambda}\,.$$

We now state a two-sided version of Hoeffding's "tail inequality." For a proof, see [35, Theorem 2].

Proposition 1. *Let* $X_1, \ldots, X_h$ *be independent random variables with sum* $Y = X_1 + \cdots + X_h$. *Assume that* $X_i \in [a_i, b_i]$. *Then, for all* $t > 0$,

$$\Pr\left(|Y - E[Y]| \geq t\right) \;\leq\; 2\,\exp\left(\frac{-2t^2}{\sum_{i=1}^{h}(b_i - a_i)^2}\right)\,.$$

5 Lower bounds on $D(n)$ and $\mathcal{R}(n)$

In this section we prove several lower bounds on $D(n)$ and $\mathcal{R}(n)$, where $n = h + d$ and h is the order of a Hadamard matrix. Theorem 1 applies when $h \geq h_0(d)$ is sufficiently large. If we assume the Hadamard conjecture, then we can drop the "sufficiently large" restriction (see Corollary 1).

If the Hadamard conjecture is false then it is sometimes necessary to take $d \geq 4$. In this case Theorems 2 and 3 are preferable as they impose weaker restrictions on h than does Theorem 1, at the cost of a slight weakening of the lower bound on $D(n)$. The proofs of Theorems 2 and 3 use Lemma 10 and Proposition 1, which are not needed for the proof of Theorem 1.

Theorem 1. *If* $d \geq 1$*,* $h \in \mathcal{H}$*,* $h \geq 4$*,* $n = h + d$*, and*

$$h \geq h_0(d) := \left(e(\pi/2)^{d/2}(d-1)! + d\right)^2, \tag{4}$$

then

$$\frac{D(n)}{h^{h/2}} > \left(\frac{2n}{\pi}\right)^{d/2}. \tag{5}$$

Proof. Let A be a Hadamard matrix of order h. We add a border of d rows and columns to give a larger matrix $\widetilde{A}$ of order n. The border is defined by matrices B, C and D as in §3. The matrices A, B, C, and D all have entries drawn from $\{\pm 1\}$. We show that a suitable choice of B, C and D gives a matrix $D - CA^{-1}B$ with sufficiently large determinant that the result can be deduced from Lemma 2.

Define $M = F - D$, where $F = CA^{-1}B$. Thus $-M$ is the Schur complement of A in $\widetilde{A}$. Note that, since A is a Hadamard matrix, $A^T = hA^{-1}$.

Following Best's approach, B is allowed to range over the set $S(h,d)$ of all $h \times d$ $\{\pm 1\}$-matrices. We give a lower bound on the mean value $\mu := E(\det(M))$ and deduce that a matrix B exists for which $\det(M) \geq \mu$. We use $E(\cdots)$ to denote a mean value over all possible choices of $B \in S(h,d)$, unless the mean value over some subset of $S(h,d)$ is specified.

The $d \times h$ matrix $C = (c_{ij})$ depends on B. We choose

$$c_{ij} = \mathrm{sgn}(B^T A)_{ij}\,,$$

where

$$\mathrm{sgn}(x) := \begin{cases} +1 \text{ if } x \geq 0, \\ -1 \text{ if } x < 0. \end{cases}$$

[*Remark.* The choice of C ensures that there is no cancellation in the inner products defining the diagonal entries of $hF = C \cdot (A^T B)$. Thus, we expect the diagonal entries f_{ii} of F to be nonnegative and of order $h^{1/2}$, but the off-diagonal entries f_{ij} $(i \neq j)$ to be of order unity with high probability.]

Best [8, Theorem 1] shows[4], using the Cauchy-Schwarz inequality, that $0 \leq f_{ii} \leq h^{1/2}$, and it follows similarly that $|f_{ij}| \leq h^{1/2}$.

[4] In [29, footnote on pg. 68] this result is attributed to J. H. Lindsey. The upper bound can be achieved infinitely often, in fact whenever a regular Hadamard matrix of order h exists. For example, this is true if $h = 4q^2$, where q is an odd prime power and $q \not\equiv 7 \pmod 8$, see [63].

We take $D = (d_{ij})_{1\le i,j\le d}$ to be a $d \times d$ matrix with diagonal entries $d_{ii} = -1$ and off-diagonal entries to be specified later.

Let $g(h)$ be as in Lemma 4. Observe that

$$E(f_{ij}) = \begin{cases} g(h) - 1 \text{ if } i = j, \\ 0 \text{ otherwise,} \end{cases}$$

where the case $i = j$ follows from Best [8, Theorem 3]. We now show that

$$E(f_{ij}^2) = 1 \;\text{ if }\; i \neq j. \tag{6}$$

To prove this, assume without essential loss of generality that $i = 1$, $j > 1$. Write $F = UB$, where $U = CA^{-1} = h^{-1}CA^T$. Now

$$f_{1j} = \sum_k u_{1k} b_{kj},$$

where

$$u_{1k} = \frac{1}{h} \sum_\ell c_{1\ell} a_{k\ell}$$

and

$$c_{1\ell} = \operatorname{sgn}\left(\sum_m b_{m1} a_{m\ell} \right).$$

Observe that $c_{1\ell}$ and u_{1k} depend only on the first column of B. Thus, f_{1j} depends only on the first and j-th columns of B. If we fix the first column of B and take expectations over all choices of the other columns, we obtain

$$E(f_{1j}^2) = E\left(\sum_k \sum_\ell u_{1k} u_{1\ell} b_{kj} b_{\ell j} \right).$$

The expectation of the terms with $k \neq \ell$ vanishes, and the expectation of the terms with $k = \ell$ is $\sum_k u_{1k}^2$. Thus, (6) follows from Lemma 7.

Now suppose that $i \neq j$, $k \neq \ell$. We cannot assume that f_{ij} and $f_{k\ell}$ are independent[5]. However, from the Cauchy-Schwarz inequality, we have

$$E(|f_{ij} f_{k\ell}|) \le \sqrt{E(f_{ij}^2) E(f_{k\ell}^2)} = 1. \tag{7}$$

[5]For example, f_{12} and f_{21} are not independent. Since f_{ij} depends on columns i and j of B, we see that f_{ij} and $f_{k\ell}$ are independent iff $\{i,j\} \cap \{k,\ell\} = \emptyset$.

Since f_{ii} depends only on the i-th column of B, the "diagonal" terms f_{ii} are independent; similarly the diagonal terms m_{ii} are independent. Now $E(m_{ii}) = E(f_{ii}) - d_{ii} = g(h)$ by our choice $d_{ii} = -1$, so

$$E\left(\prod_{i=1}^{d} m_{ii}\right) = \prod_{i=1}^{d} E(m_{ii}) = g(h)^d.$$

Observe that $\det(F+I)$ is the sum of a diagonal term $\prod_{1\le i\le d} m_{ii}$ and $(d!-1)$ other terms. If $d > 1$, the latter terms each contain at most $d-2$ factors of the form m_{ii} (bounded by $h^{1/2}+1$) and at least two factors of the form f_{ij}. The expectations of the latter terms are bounded by $(h^{1/2}+1)^{d-2}$. For example, if $d = 3$, we use

$$|E(f_{12}f_{21}m_{33})| \le E(|f_{12}f_{21}|)\max(|m_{33}|) \le h^{1/2}+1.$$

In general, we use an upper bound $h^{1/2}+1$ for $d-2$ of the factors, and save a factor of order h by using (7) once.

Thus

$$E(\det(F+I)) \ge g(h)^d - (d!-1)\,(h^{1/2}+1)^{d-2}. \tag{8}$$

We simplify (8) using $h^{1/2}+1 \le h^{1/2}\exp(h^{-1/2})$ and, from (4), $d < h^{1/2}$. Thus $(h^{1/2}+1)^{d-2} \le h^{d/2-1}\exp(dh^{-1/2}) \le h^{d/2-1}e$, and (8) gives

$$E(\det(F+I)) \ge g(h)^d - d!\,h^{d/2-1}e. \tag{9}$$

Now, using Lemma 4 gives

$$E(\det(F+I)) > (ch^{1/2}+0.9)^d - d!\,h^{d/2-1}e \ge c^d h^{d/2}\left(1 + \frac{0.9d}{ch^{1/2}} - \frac{d!e}{c^d h}\right). \tag{10}$$

We also have

$$\left(\frac{h}{n}\right)^{d/2} = \left(1+\frac{d}{h}\right)^{-d/2} > \exp\left(-\frac{d^2}{2h}\right). \tag{11}$$

Now $h \ge h_0(d)$ implies both $d^2 < h$ and $dh^{1/2} \ge d!e/c^d + d^2$; since $c < 0.9$ the latter inequality implies

$$\frac{0.9d}{ch^{1/2}} > \frac{d!e}{c^d h} + \frac{d^2}{h}. \tag{12}$$

From $d^2 \le h$ and the inequalities (10)–(12), we have

$$\begin{aligned} E(\det(F+I)) &> c^d n^{d/2}\left(1 + \frac{0.9d}{ch^{1/2}} - \frac{d!e}{c^d h}\right)\exp\left(-\frac{d^2}{2h}\right) \\ &> c^d n^{d/2}\left(1 + \frac{d^2}{h}\right)\exp\left(-\frac{d^2}{2h}\right) > c^d n^{d/2}. \end{aligned}$$

This proves the existence of matrices B and C such that $\det(F+I) > c^d n^{d/2}$.

To complete the proof, we choose the off-diagonal elements of D, in an arbitrary order, in such a manner that $\det(F-D) \ge \det(F+I)$. This is always possible, since $\det(F-D)$ is a linear function of each off-diagonal element d_{ij} considered separately, so at least one of the choices $d_{ij} = +1$ and $d_{ij} = -1$ does not reduce the determinant. The inequality (5) now follows from Lemma 2. $\square$

Remark 4. A variant of Theorem 1 arises if we start, not from an $h \times h$ Hadamard matrix, but from an $h \times h$ *conference* matrix[6], that is a $\{0, \pm 1\}$-matrix C, with $\mathrm{diag}(C) = 0$, satisfying $CC^T = (h-1)I$. To prove the variant, we need only minor alterations to Lemma 7 and to the proof of Theorem 1. Using this variant, we can improve the constant[7] in Theorem C of Neubauer and Radcliffe [45] from 0.3409 to 0.4484. Another interesting variant allows all matrices to have entries from the set $\{\pm 1, \pm i\}$; then a 4-sided "coin" and 4-valued "sign" function need to be used.

Corollary 1. *If $1 \le d \le 3$, $h \in \mathcal{H}$, $h \ge 4$, and $n = h + d$, then*

$$\frac{D(n)}{h^{h/2}} > \left(\frac{2n}{\pi}\right)^{d/2} \tag{13}$$

and

$$\mathcal{R}(n) > \left(\frac{2}{\pi e}\right)^{d/2}. \tag{14}$$

Proof. First consider the inequality (13). This follows from Theorem 1 if $h \ge h_0(d)$. The inequality (8) in the proof of Theorem 1 covers all cases with

[6]Similarly for weighing matrices [19], which are also scalar multiples of orthogonal matrices.

[7]This constant occurs in the statement that the Ehlich upper bound [27] for $D(n)$ in the case $n \equiv 3 \bmod 4$ is attained *up to a constant factor* infinitely often.

$h \ge 16$ and $d \le 3$, so we only need check the cases $h \in \{4, 8, 12\}$ and use the known values (see for example [48]) of $D(5), \ldots, D(15)$.

The inequality (14) follows from (13) and Lemma 5 (with $\alpha = d$). $\square$

Remark 5. If the Hadamard conjecture is true, then for $4 < n \not\equiv 0 \pmod 4$, we can take $h = 4\lfloor n/4 \rfloor$ and $d = n - h \le 3$ in Corollary 1. Thus,

$$1 > \mathcal{R}(n) > \left(\frac{2}{\pi e}\right)^{d/2} \ge \left(\frac{2}{\pi e}\right)^{3/2} > 0.1133\,.$$

The following corollary does not assume the Hadamard conjecture, but it does require h to be sufficiently large.

Corollary 2. *Assume that $d > 0$, $h \in \mathcal{H}$, and $h \ge h_0(d)$, where $h_0(d)$ is as in Theorem 1. If $n = h + d$, then*

$$\mathcal{R}(n) > \left(\frac{2}{\pi e}\right)^{d/2}.$$

Proof. This follows from Theorem 1 and Lemma 5 (with $\alpha = d$). $\square$

Corollary 3. *Let $d \ge 0$ be fixed. Then*

$$\liminf_{\substack{n\to\infty \\ n-d\in\mathcal{H}}} \mathcal{R}(n) \ge \left(\frac{2}{\pi e}\right)^{d/2}.$$

Proof. The result is trivial if $d = 0$, so suppose that $d \ge 1$. Corollary 2 shows that $\mathcal{R}(n) > (2/(\pi e))^{d/2}$ for $n = h + d$ and all sufficiently large h, so the result follows. $\square$

Corollary 4. *There exist positive constants κ_d such that, if $d \ge 0$, $h \ge 4$, $h \in \mathcal{H}$, and $n = h + d$, then $\mathcal{R}(n) \ge \kappa_d$.*

Proof. The result is trivial if $d = 0$. Otherwise, define

$$\kappa_d := \min\,\{(2/(\pi e))^{d/2}\} \cup \{\mathcal{R}(n) \,|\, n \in \mathbb{N},\, n - d \in \mathcal{H},\, 4 \le n - d < h_0(d)\}\,.$$

Since κ_d is the minimum of a finite set of positive values, it is positive, and by Corollary 2 it is a lower bound on $\mathcal{R}(n)$. $\square$

Remark 6. The best (i.e. largest) possible values of the constants κ_d are unknown, except for the trivial $\kappa_0 = 1$. From Corollary 1, we know that

$$\kappa_d \geq \left(\frac{2}{\pi e}\right)^{d/2} \tag{15}$$

holds for $d \leq 3$, and it is plausible to conjecture that (15) holds for all $d \geq 0$. It is unlikely that this inequality is tight, and plausible that the constant $2/(\pi e)$ could be replaced by some greater value.

If the Hadamard conjecture is true, then we can assume that $d \leq 3$ and $\kappa_d \geq (2/(\pi e))^{3/2} > 1/9$. Hence, it is of interest to mention known upper bounds on the κ_d for $d \leq 3$.

1. We have $\kappa_1 \leq \mathcal{R}(9) = 7 \times 2^{11}/3^9 < 0.7284$, which is sharper than the value $(2/e)^{1/2} \approx 0.8578$ given by the Barba bound [5] as $n \to \infty$.

2. The Ehlich-Wojtas bound [26, 62] in the limit as $n \to \infty$ shows that $\kappa_2 \leq 2/e < 0.7358$.

3. We have $\kappa_3 \leq \mathcal{R}(11) = 5 \times 2^{16}/11^{11/2} < 0.6135$, which is sharper than the value $2e^{-3/2}11^3 7^{-7/2} \approx 0.6545$ given by Ehlich's upper bound [27] as $n \to \infty$.

We now state and prove three similar theorems. In the proofs of Theorems 2 and 3 we need the Schur matrix F to have off-diagonal entries small compared to its diagonal entries so that we can apply the determinant bound for diagonally-dominant matrices in Lemma 10. To quantify this we introduce two sets S_0 and S_1. Roughly speaking, S_0 is the set of coin-tosses yielding large-enough diagonal entries of F, and S_1 is the set of coin-tosses yielding too-large off-diagonal entries of F. It is necessary to show that $S_0 \backslash S_1 \neq \emptyset$. We accomplish this by using Lemma 12 and an independence argument to show that, with our choice of parameters, S_0 is not too small and (by using a Hoeffding tail bound) S_1 is smaller than S_0. The two theorems differ in the choice of parameters and largeness/smallness criteria. Theorem 2 gives the sharper bound but has more restrictive conditions, in particular the condition $h \geq 16d^3 \ln h$. Theorem 3 relaxes this condition to $h \geq 6d^3$, but at the cost of a weaker bound on $\mathcal{R}(n)$. Finally, Theorem 4 removes any restriction on h, at the cost of a yet weaker bound (but still depending only on d).

Theorem 2. *Let $d \geq 0$ be given, and let $h \in \mathcal{H}$, $h \geq 656$, be such that*

$$16d^3 \leq \frac{h}{\ln h} \,. \tag{16}$$

If $n = h + d$ and

$$\varepsilon = \left(\frac{4d \ln h}{h}\right)^{1/2} , \tag{17}$$

then

$$\frac{D(n)}{h^{n/2}} \geq \left(\frac{2}{\pi}\right)^{d/2} \exp(-2.31\, d\varepsilon) \,. \tag{18}$$

Note that when $d \to \infty$ or $h \to \infty$ then (16)–(17) imply that $\varepsilon \to 0+$. Before proving Theorem 2, we state a lemma which collects some of the inequalities that are required.

Lemma 13. *Under the conditions of Theorem 2, if $d \geq 1$ then the following six inequalities hold:*

$$d\varepsilon \leq 1/2 \,, \tag{19}$$

$$\varepsilon \geq 8d/h \,, \tag{20}$$

$$\varepsilon \leq \frac{(2/\pi)^{1/2} - 0.5}{1.1} \approx 0.2704 \,, \tag{21}$$

$$2d^2 \exp(-\varepsilon^2 h/8) \leq (2\varepsilon)^d \,. \tag{22}$$

$$1 - \frac{1.1\varepsilon}{c} \geq \exp(-\alpha\varepsilon), \tag{23}$$

$$g(h) - 1 \geq ((2/\pi)^{1/2} - \varepsilon/10)h^{1/2}, \tag{24}$$

where $\alpha \approx 1.7262$, $c = \sqrt{2/\pi}$, and $g(h)$ is as in Lemma 4.

Proof. From (16) and (17) we have

$$d^2\varepsilon^2 = \frac{4d^3 \ln h}{h} \leq \frac{1}{4} \,,$$

which proves (19). For (20) use $\ln h \geq 1$. Thus, from (16), $h \geq 16d^3 \geq 16d$, so

$$\varepsilon^2 = \frac{4d \ln h}{h} \geq \frac{4d}{h} \geq \frac{64d^2}{h^2} \,,$$

and taking a square root gives (20). Similarly, using (16) and (17) gives

$$\varepsilon \le \left(\frac{2\ln h}{h}\right)^{1/3},$$

and the condition $h \ge 656$ then gives $\varepsilon \le ((2\ln 656)/656)^{1/3} \approx 0.2704$, which proves (21).

Taking logarithms shows that the inequality (22) is equivalent to $\varepsilon^2 h/8 \ge \ln(2d^2) - d\ln(2\varepsilon)$, and substituting the definition (17) of ε and simplifying shows that this is equivalent to

$$\ln(16d\ln h) \ge 2\ln(2d^2)/d\,. \tag{25}$$

The right side of (25) is bounded above by $4\sqrt{2}/e \approx 2.081$, but the left side exceeds this value for all $d \ge 1$ and $h \ge 2$. This completes the proof of (22).

To show (23), recall that $\varepsilon \le 0.271$. Using Lemma 11 with $\varepsilon_0 = 0.271$, $\kappa = -1.1/c$, we see that (23) is valid for $\alpha \ge -\ln(1-0.271\times 1.1/c)/0.271 \approx 1.7262$.

Finally, for (24), Lemma 4 gives $g(h) > ch^{1/2} + 0.9$. Thus, it is sufficient to show that $ch^{1/2}+0.9-1 \ge (c-\varepsilon/10)h^{1/2}$, which is equivalent to $\varepsilon h^{1/2} \ge 1$. This follows easily from (17). □

Proof of Theorem 2. As usual, we can assume that $d \ge 1$, as the result is trivial if $d = 0$. We use the same notation as in the proof of Theorem 1. In particular, $c = \sqrt{2/\pi}$, $F = CA^{-1}B = h^{-1}CA^TB$, and $M = F - D$, where $\mathrm{diag}(D) = -I$.

Consider f_{ij} for i fixed and $j \ne i$. To simplify the notation, assume that $i = 1$ and $j \ne 1$. Then

$$f_{1j} = \frac{1}{h}\sum_k\sum_\ell c_{1k}a_{\ell k}b_{\ell j} = \sum_\ell u_{1\ell}b_{\ell j} \quad \text{say,}$$

where

$$u_{1\ell} = \frac{1}{h}\sum_k c_{1k}a_{\ell k} \tag{26}$$

and

$$c_{1k} = \mathrm{sgn}(B^TA)_{1k} = \mathrm{sgn}\left(\sum_\ell b_{\ell 1}a_{\ell k}\right).$$

We see that c_{1k} depends on column 1 of B and is independent of the other columns of B. Thus, f_{1j} depends on columns 1 and j of B and is independent of the other columns of B. Also, from Lemma 7,

$$\sum_{\ell} u_{1\ell}^2 = 1. \tag{27}$$

Consider fixing the first column of B and allowing the other columns to vary uniformly at random. Thus, for fixed $j \in [2, d]$, we can regard $X_\ell := u_{1\ell} b_{\ell j}$, $1 \le \ell \le h$, as h independent random variables having expectation zero and sum f_{1j}. Also, $|X_\ell| \le |u_{1\ell}|$. Thus, by (27) and Proposition 1, we have

$$\Pr\left(|f_{1j}| \ge t\right) \le 2e^{-t^2/2} \quad \text{for} \quad t > 0. \tag{28}$$

The inequality (28) is valid for any choice of the first column of B, hence it is valid if the first column is chosen at random. Now allow all columns of B to vary uniformly at random. Since there are $d(d-1)$ off-diagonal elements f_{ij}, it follows (without assuming independence of the f_{ij}) that[8]

$$\Pr\left(\max_{i \ne j} |f_{ij}| \ \ge \ t\right) \le 2d(d-1)e^{-t^2/2}. \tag{29}$$

[*Remark*: The inequality (29) shows that the off-diagonal elements of F are usually "small", more precisely of order $\sqrt{\log(d+1)}$. We now consider the diagonal elements and show that there is a set (not too small) on which they are at least $h^{1/2}/2$.]

As in the proof of Theorem 1 (following Best [8, Theorem 3]),

$$E(f_{ii}) = g(h) - 1,$$

where $g(h) \sim ch^{1/2}$ is as in Lemma 4. Choose $c_1 < c$ and suppose that h is sufficiently large that $E(f_{ii}) = g(h) - 1 \ge c_1 h^{1/2}$.

Choose $c_2 < c_1$, and consider $\rho_i := \Pr\left(f_{ii} \ge c_2 h^{1/2}\right)$. By our choice of C and Best [8, Thm. 1], we have $0 \le f_{ii} \le h^{1/2}$. Thus, by Lemma 12 applied to the random variable $f_{ii}/h^{1/2}$, we have

$$\rho_i \ge \frac{c_1 - c_2}{1 - c_2}\,.$$

[8] We could sharpen the argument at this point by using the Lovász Local Lemma [28] to reduce the right-hand-side of (29) to $O(de^{-t^2/2})$, but this would not significantly improve the final bound (18).

Note that f_{ii} depends only on the i-th column of B, so the f_{ii} are independent for $1 \le i \le d$. Thus, if $S_0 = \{B \mid \min\{f_{ii} | 1 \le i \le d\} \ge c_2 h^{1/2}\}$, we have

$$\Pr(S_0) = \prod_i \rho_i \ge \left(\frac{c_1 - c_2}{1 - c_2}\right)^d .$$

To be definite take $c_1 = c - \varepsilon/10$ and $c_2 = c_1 - \varepsilon$, where ε is as in the statement of the theorem and, from Lemma 13, $\varepsilon \le (c - 0.5)/1.1 \approx 0.2704$. Then we have $c_2 = c - 1.1\varepsilon \ge 1/2$, $\rho_i \ge 2\varepsilon$, and $\Pr(S_0) \ge (2\varepsilon)^d$.

Let S_1 be the set of B for which $\max_{i \ne j} |f_{ij}| \ \ge \ t$. From (29), we have $\Pr(S_1) \le 2d(d-1)e^{-t^2/2}$. For the matrix F to be $\mathrm{DD}(\varepsilon)$ on a nonempty set $S_0 \backslash S_1$ of choices of B, it suffices that

$$t \le c_2 \varepsilon h^{1/2} \quad \text{and} \quad 2d(d-1)e^{-t^2/2} < (2\varepsilon)^d . \tag{30}$$

Thus, choosing $t = c_2 \varepsilon h^{1/2}$, it is sufficient that

$$2d^2 \exp(-c_2^2 \varepsilon^2 h/2) \le (2\varepsilon)^d. \tag{31}$$

Since $c_2 \ge 1/2$, part (22) of Lemma 13 shows that the inequality (31) is satisfied. Thus, Lemma 10 applied to F gives

$$\det(F) \ge (c_2 h^{1/2})^d (1 - (d-1)^2 \varepsilon^2) \tag{32}$$

on a nonempty set $S_0 \backslash S_1$. Since $d\varepsilon \le 1/2$,

$$1 - (d-1)^2 \varepsilon^2 \ge 1 - d^2 \varepsilon^2 \ge 1 - d\varepsilon/2 \ge \exp(-\beta d \varepsilon),$$

where Lemma 11 gives $\beta = 2\ln(4/3) \approx 0.5755$. As in the proof of Theorem 1, we choose the elements of D so that $\det(M) = \det(F - D) \ge \det(F)$. It follows from Lemma 2 that

$$D(n) \ge h^{n/2} c_2^d \exp(-\beta d \varepsilon). \tag{33}$$

To complete the proof, use (23) of Lemma 13. We have $c_2/c \ge \exp(-\alpha\varepsilon)$, where $\alpha \approx 1.7262$, and

$$c_2^d \ge c^d \exp(-\alpha d \varepsilon). \tag{34}$$

Now the theorem follows from (33), using $\alpha + \beta < 2.31$. $\square$

The inequality in the following Corollary 5 is slightly weaker than the inequality in Corollary 2, but Corollary 5 is applicable for smaller values of h. Note that $d\varepsilon \le \frac{1}{2}$ from (19), so $\exp(-2.38d\varepsilon) \ge \exp(-1.19) > 0.3$.

Corollary 5. *Under the conditions of Theorem* 2*,*

$$\mathcal{R}(n) > \left(\frac{2}{\pi e}\right)^{d/2} \exp(-2.38\, d\varepsilon).$$

Proof. We can assume that $d > 0$. From Lemma 6 with $\alpha = d$,

$$\left(\frac{h}{n}\right)^n \;>\; \exp(-d)\exp(-d^2/h).$$

From (20) of Lemma 13, $d^2/h \le d\varepsilon/8$. Thus

$$(h/n)^{n/2} \;>\; \exp(-d/2)\exp(-d\varepsilon/16). \tag{35}$$

The result now follows from (18) and (35), since $2.31 + 1/16 < 2.38$. $\square$

Theorem 3 weakens the condition (16) on h in Theorem 2 by eliminating the log term; the new condition is $h \ge 6d^3$. The cost is a weakening of the result – essentially the constant $2/\pi$ in inequality (18) is replaced by a smaller constant, and we have to introduce a factor $1 - O(d^3/h)$.

Theorem 3. *Let* $\delta = 6d^3/h$*, and assume that* $\delta \le 1$*,* $d > 0$*,* $h \in \mathcal{H}$*, and* $n = h + d$*. Then*

$$\frac{D(n)}{h^{n/2}} \;\ge\; (0.594)^d\,(1 - 0.93\,\delta)$$

and

$$\mathcal{R}(n) \;\ge\; (0.352)^d\,(1 - 0.93\,\delta) \;\ge\; 0.07\,(0.352)^d\,.$$

Proof. We follow the notation and proof of Theorem 2, but with a different choice of c_1, c_2 and ε.

If $d \le 3$ the results follow from Corollary 1, so assume that $d \ge 4$. Since $h \ge 6d^3$, we can assume that $h \ge 384$.

Choose $c_1 = c(1 - 1/(4h))$. By Lemma 4, $g(h) - 1 > c_1 h^{1/2}$. Since $h \ge 384$, we have $c_1 \ge 0.797$. Now choose $c_2 = 2c_1 - 1 \ge 0.594$ so that $\rho_i \ge (c_1 - c_2)/(1 - c_2) \ge 1/2$ and $\Pr(S_0) \ge 2^{-d}$.

For the matrix F to be $\mathrm{DD}(\varepsilon)$ on a nonempty set $S_0 \backslash S_1$ of choices of B, it suffices that

$$t \le c_2 \varepsilon h^{1/2} \quad \text{and} \quad 2d(d-1)e^{-t^2/2} < 2^{-d}\,.$$

Thus, choosing $t = c_2 \varepsilon h^{1/2}$, it is sufficient that

$$2d(d-1)\exp(-c_2^2 \varepsilon^2 h/2) < 2^{-d}\,,$$

which is equivalent to

$$\varepsilon^2 > \frac{2d\ln 2}{c_2^2 h}\left(1 + \frac{\log_2(2d(d-1))}{d}\right)\,. \tag{36}$$

Now $2\ln 2/c_2^2 < 3.92$, and (36) is satisfied if we choose ε so that

$$\varepsilon^2 = \frac{3.92d}{h}\left(1 + \frac{\log_2(2d(d-1))}{d}\right)\,.$$

To obtain a nontrivial bound from Lemma 10 we need $(d-1)^2\varepsilon^2 < 1$, or equivalently

$$\frac{3.92d(d-1)^2}{h}\left(1 + \frac{\log_2(2d(d-1))}{d}\right) \;<\; 1.$$

We find numerically[9] that

$$\max_{d\in\mathbb{N},\, d\ge 4}\left[\frac{3.92(d-1)^2}{d^2}\left(1 + \frac{\log_2(2d(d-1))}{d}\right)\right] \;<\; 5.57.$$

Thus, the condition $h \ge 6d^3$ is sufficient for F to be $\mathrm{DD}(\varepsilon)$ on a nonempty set. Also, we have $(d-1)^2\varepsilon^2 < 5.57\,\delta/6 < 0.93\,\delta$, so $1-(d-1)^2\varepsilon^2 > 1 - 0.93\,\delta$. Now (32) and the remainder of the proof follow as in the proof of Theorem 2, using Lemma 6 with $\alpha = d$ for the inequality involving $\mathcal{R}$, and observing that

$$\frac{d^2}{2h} = \frac{d^3}{2h}\cdot\frac{1}{d} \le \frac{1}{12d} \le \frac{1}{48}$$

and

$$0.594\exp\left(-\frac{1}{2} - \frac{1}{48}\right) > 0.352.$$

□

[9]The maximum $5.564\ldots$ occurs at $d = 9$.

We now investigate when the conditions of Theorem 3 are satisfied. First we state a result of Livinskyi [41, Theorem 5.4]. This result is better for our purposes than the (asymptotically sharper) result of Livinskyi quoted in §2, as it has a smaller additive constant.

Proposition 2 (Livinskyi, Theorem 5.4)**.** *If p is an odd positive integer and $t = 6\lfloor \frac{1}{26} \log_2 \left(\frac{p-1}{2}\right) \rfloor + 11$, then there exists a Hadamard matrix of order $2^t p$.*

Corollary 6. *If $k \in \mathbb{N}$, $q \in \mathbb{N}$, and $1 \le q \le 2^{26k+1}$, then there exists a Hadamard matrix of order $2^{6k+5}q$.*

Proof. If p is odd and $0 \le (p-1)/2 < 2^{26k}$, then Proposition 2 shows that $2^{6k+5}p \in \mathcal{H}$. Thus, if $q = 2^m p$ where p is odd, the Sylvester construction applied m times shows that $2^{6k+5}q \in \mathcal{H}$. $\square$

Lemma 14. *If $h_i \in \mathcal{H}$, $h_{i+1} \in \mathcal{H}$ are consecutive Hadamard orders and $h_i \ge 3 \times 2^{70}$, then $6(h_{i+1} - h_i)^3 \le h_i$.*

Proof. From Corollary 6 with $k \ge 3$, the gaps between consecutive Hadamard orders $h_i, h_{i+1} \le 2^{32k+6}$ are at most 2^{6k+5}, and $6 \times (2^{6k+5})^3 = 3 \times 2^{18k+16}$, so the result holds for $h_i, h_{i+1} \in I_k := [3 \times 2^{18k+16}, 2^{32k+6}]$. Now the intervals $I_3 = [3 \times 2^{70}, 2^{102}]$, $I_4 = [3 \times 2^{78}, 2^{134}], \ldots,$ overlap and cover the whole region $[3 \times 2^{70}, \infty)$. Also, $I_k \cap I_{k+1} = [3 \times 2^{18k+34}, 2^{32k+6}]$ is sufficiently large that the special case $h_i \in I_k$, $h_{i+1} \in I_{k+1}$ causes no problem, as $h_{i+1} - h_i \le 2^{6(k+1)+5} = 2^{6k+11}$ and both of h_i, h_{i+1} must belong to one of I_k or I_{k+1}. $\square$

The following lemma shows that the condition $\delta \le 1$ (that is $6d^3 \le h$) of Theorem 3 is always satisfied for n sufficiently large.

Lemma 15. *Suppose $n \in \mathbb{N}$, $n \ge 60480$, $h = \max\{x \in \mathcal{H} \mid x \le n\}$, and $d = n - h$. Then $6d^3 \le h$.*

Sketch of proof. The proof is mainly based on machine computations, so we can only give an outline here. We split the interval $[60480, \infty)$ into several sub-intervals and consider each such sub-interval separately. We choose a set of intervals that overlap slightly in order to avoid any difficulties near the boundaries between adjacent intervals. (Discussion of such minor details is omitted below.)

First consider $[60480, 2^{31}]$. We wrote a C program to list a subset L of the known Hadamard orders $h \le 2^{31}$ using several (by no means all) known constructions [1, 6, 7, 20, 22, 24, 25, 33, 34, 36, 38, 43, 49, 52, 57, 58, 61, 64]. The constructions that we used were:

1. Paley-Sylvester-Turyn: if p is prime (or $p = 0$) and $j, k \geq 0$ are integers, then $h = 2^j(p^k + 1) \in \mathcal{H}$ whenever $4|h$.

2. Agaian-Sarukhanyan: if $\{4a, 4b\} \subset \mathcal{H}$, then $8ab \in \mathcal{H}$.

3. Craigen-Seberry-Zhang: if $\{4a, 4b, 4c, 4d\} \subset \mathcal{H}$, then $16abcd \in \mathcal{H}$.

4. Twin-Prime construction: if q and $q + 2$ are both odd prime powers, then $h = (q + 2)q + 1 \in \mathcal{H}$.

5. Craigen-Holzmann-Kharaghani [20, Cor. 16, pg. 87]: If $q = x + y$ is a sum of two complex Golay numbers x and y, then $h = 8q \in \mathcal{H}$. It is known that every integer g of the form $g = 2^{a-1}\,6^b\,10^c\,22^d\,26^e$, with $a, b, c, d, e \geq 0$ integers, is complex Golay. For example: 659×8, 739×16, 971×8, and 1223×16 are all in $\mathcal{H}$ since $659 = 11 + 648$, $739 \times 2 = 26 + 1452$, $971 = 968 + 3$, and $1223 \times 2 = 26 + 2420$.

6. Miyamoto-I: if $q - 1 \in \mathcal{H}$ and q is a prime power, then $4q \in \mathcal{H}$.

7. Miyamoto-II: if q and $2q - 3$ are prime powers and $q \equiv 3 \bmod 4$, then $8q \in \mathcal{H}$.

8. Yamada/Kiyasu: if q is a prime power, $q \equiv 5 \bmod 8$, and $(q+3)/2 \in \mathcal{H}$, then $4(q + 2) \in \mathcal{H}$.

9. Small orders: 1, 2, and all h divisible by 4 with $4 \leq h \leq 2056$ are in $\mathcal{H}$ *except* perhaps 668, 716, 892, 1004, 1132, 1244, 1388, 1436, 1676, 1772, 1916, 1948, 1964.

10. Baumert-Hall-Williamson: If w is the order of a quadruple of Williamson matrices, and $4b$ is the order of a Baumert-Hall array, then $4bw \in \mathcal{H}$. Known Williamson numbers include all w with $1 \leq w \leq 64$ *except* $\{35, 47, 53, 59\}$. Known Baumert-Hall numbers b include all b with $1 \leq b \leq 108$ *except* $\{97, 103\}$, and all $b = 2^k + 1$ for $k \geq 0$.

11. Seberry-Yamada [52, Cor. 29]: If q and $2q + 3$ both are prime powers, then $w = 2q+3$ is a Williamson order. For example, 109 is a Williamson order.

Using the computed L it is easy to check if any given $n \leq 2^{31}$ corresponds to a pair (h, d) (with h, d defined as in the statement of the lemma) such

that $6d^3 > h$. We found that the largest such n is 60480, corresponding to the open interval $(60456, 60480)$ which does not intersect our list L of known Hadamard orders (and $6 \times 23^3 = 73002 > 60456$). Thus, we have proved the result claimed for $n \le 2^{31}$.

Now consider the interval $(2\times 10^9, 8\times 10^{18}]$. There is some overlap with the previous case, since $2 \times 10^9 < 2^{31}$. We use the tables of maximal prime gaps at [46, 54] (found by e Silva and others) for primes $p \le 4 \times 10^{18}$. The largest of these prime gaps is 1476. Using the tables and the fact that $2(p+1) \in \mathcal{H}$ for every odd prime p, we find that the claim holds for $2\times 10^9 < n \le 8\times 10^{18}$.

The tables of maximal prime gaps do not yet extend as far as 2×10^{21}. Hence we deal with the interval $(8\times 10^{18}, 4\times 10^{21}]$ in a different manner, but still using the tables of known maximal prime gaps.

First consider the interval $(7 \times 10^{18}, 1.2 \times 10^{20}]$. Since $32(p+1) \in \mathcal{H}$ for prime p, it is sufficient to know prime gaps for primes $p \le 1.2 \times 10^{20}/32 < 4\times 10^{18}$. The largest such prime gap is 1476, corresponding to a gap between Hadamard orders of at most $32\times 1476 = 47232$. Since $6\times 47232^3 < 7\times 10^{18}$, the claim holds for $7 \times 10^{18} < n \le 1.2 \times 10^{20}$.

Now consider the interval $(10^{20}, 4 \times 10^{21}]$. Since $1000(p+1) \in \mathcal{H}$ for prime p, the known prime gaps for $p \le 4 \times 10^{18}$ suffice. The largest such gap, 1476, now corresponds to a gap between Hadamard orders of at most 1476000. Since $6\times 1476000^3 < 10^{20}$, the claim holds for $10^{20} < n \le 4\times 10^{21}$.

Finally, since $3 \times 2^{70} < 4 \times 10^{21}$, Lemma 14 shows that the claim holds for all $n > 4 \times 10^{21}$, which completes the proof. $\square$

By considering a small set of exceptional cases, we now show that the condition $6d^3 \le h$ of Theorem 3 can be dropped entirely, if we are satisfied with a slightly weaker lower bound on $\mathcal{R}(n)$.

Theorem 4. *Suppose that $n \in \mathbb{N}$, $h = \max\{x \in \mathcal{H} \mid x \le n\}$, and $d = n - h$. Then*

$$\mathcal{R}(n) > 0.07\,(0.352)^d > 3^{-(d+3)}.$$

Proof. For $0 \le d \le 3$, the result follows from Corollary 1. This covers all $n < 668$. On the other hand, if $n \ge 60480$, the result follows from Theorem 3 and Lemma 15. Thus, we can assume that $668 \le n < 60480$ and $d \ge 4$.

If $n + 1 \in \mathcal{H}$ then Theorem 1 of [10] gives $\mathcal{R}(n) \ge (4/(ne))^{1/2}$, and for $n < 60480$, $d \ge 4$, it is easy to verify that $(4/(ne))^{1/2} > 0.002 > 0.07\,(0.352)^d$. Thus, if h, h' are consecutive (known) Hadamard orders, we only have to consider the cases $n = h + d$ for $4 \le d \le h' - h - 2$.

h	h'	d	p	method
664	672	$[5, 6]$	331	Paley1
712	720	$[5, 6]$	709	conference
888	896	6	443	Paley1
1000	1008	6	499	Paley1
1128	1136	6	563	Paley1
1240	1248	6	619	Paley1
2868	2880	$[8, 10]$	1433	Paley2
5744	5760	$[10, 14]$	5749	conference
10048	10064	$[12, 14]$	5023	Paley1
23980	24000	$[16, 18]$	23993	conference
47964	47988	$[20, 22]$	47963	Paley1
53732	53760	$[21, 26]$	53731	Paley1
60456	60480	22	60457	conference

Table 1: Exceptional cases in the proof of Theorem 4.

From the output of the C program described in the proof of Lemma 15, we find that the cases that are not covered by Lemma 15 or the remarks already made are those listed in Table 1, which gives 32 cases in 13 intervals. For each of these 13 intervals $[h, h']$ we know that h and h' are Hadamard orders, but we do not *know*[10] any Hadamard orders in the open interval (h, h'), and we need to verify that the inequality

$$\mathcal{R}(n) > 0.07\,(0.352)^d \tag{37}$$

is satisfied for each $n = h + d$ and the values of d listed in the third column of the table.

Using Magma [9], we wrote a program that implements a randomised algorithm to obtain a lower bound on $\mathcal{R}(n)$. The program constructs a Hadamard matrix A of order $h = (p + 1)$ or $h = 2(p + 1)$, where p is an odd prime and in the first case $p \equiv 3 \bmod 4$, using the appropriate Paley construction [49], followed if necessary by the Sylvester construction [56]. The program then generates a border of width d to obtain a matrix $\widetilde{A}$ of order n,

[10]This may just reflect our ignorance. Certainly such orders exist if the Hadamard conjecture is true. In some cases we know that they exist via constructions that were not implemented in our C program.

as in the proof of Theorem 1, and computes $|\det(\widetilde{A})|/n^{n/2}$ by computing the determinant of the Schur complement of A in $\widetilde{A}$ and using Lemma 2. If desired, several independent random trials can be performed to improve the lower bound.

Using our Magma program with the primes p listed in the fourth column of Table 1, we were able to show that the inequality (37) holds for all the cases labelled “Paley1” or “Paley2”. In fact, a few trials of our randomised algorithm were sufficient to show that the stronger inequality

$$\mathcal{R}(n) > \left(\frac{2}{\pi e}\right)^{d/2} \tag{38}$$

holds in these cases (this is not surprising, in view of Corollaries 1 and 2).

For the intervals $[h, h']$ labelled “conference” in Table 1, there is no prime p for which $h = p + 1$ or $2(p + 1)$, but there is a prime p (given in the fourth column of the table) which can be used to construct a conference matrix of order $p+1$ close to h. Using a slight modification of our Magma program, we can use this conference matrix to obtain lower bounds on $\mathcal{R}(n)$ for $n \ge p+1$ (see Remark 4). In this way we showed that the inequality (38) holds for all the intervals labelled “conference” with the exception of the interval $[712, 720]$. Here there is no suitable prime inside the interval, so we use $p = 709 < h = 712$, thus obtaining weaker lower bounds. However, we still obtain $\mathcal{R}(712 + d) > 0.352^d$ for $d \in \{5, 6\}$ by this method, and this bound is sufficient since it is stronger than the desired inequality (37).

There is one further point to consider. We illustrate it for the interval $[5744, 5760]$ of length 16. It is *possible* that 5748, 5752 and/or 5756 are Hadamard orders (although we do not at present know how to construct Hadamard matrices of these orders). Thus, we need to check that our lower bound on $\mathcal{R}(n)$ holds for $h = 5748$, $d = 10$, $n = h + d = 5758$ (and other similar cases). The prime $p = 5749$ gives a conference matrix of order 5750. Using this conference matrix, our program shows that $\mathcal{R}(5758) > 0.002115 > (2/(\pi e))^{10/2}$, so (38) is satisfied. The other, similar, cases that arise if a Hadamard order exists in the interior of any of the intervals listed in Table 1 can be covered by one of the arguments that we have already used. Thus, the inequality (37) always holds for the exceptional cases listed in Table 1. $\square$

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